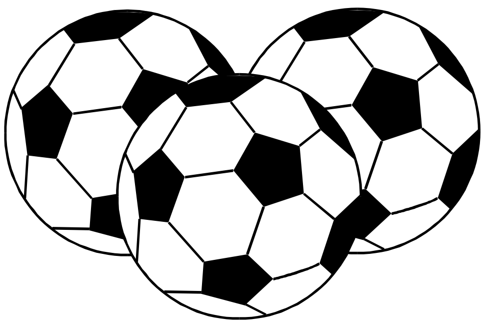
Chapter 1

Numbers

Most people can think about simple numbers without having any problems. For instance, most people (except very young kids) could count the number of soccer balls in the picture below and tell me there are three.



Everyday you count things. You might count how much money you have to see whether you can buy an ice-cream. When you count how many things there are, you are using *counting numbers* which go 0, 1, 2, 3, 4, 5, 6… The '…' symbol means "and so forth". People use '…' when they don't want to write out something obvious. In this case, this *sequence* of numbers, 0, 1, 2, 3, 4, 5, 6 keeps going on forever, so we *could* keep writing 7, 8, 9, 10, 11, 12 and so on. But instead we just write 0, 1, 2, 3, 4, 5, 6… and the '…' tells the reader, you, that this pattern keeps on going. So what types of numbers are there? We'll start with some of the simple ones.

Natural, counting and whole numbers

When you count 'how many' you are using the *natural*, *counting* or *whole* numbers. For instance the number 3 is a natural, or counting, or whole number. So is 5. So is 1. So is 125.

Some people say that 0 is also a natural, counting or whole number. Ask your teacher what they think – your school may say it is or isn't. It sort of makes sense – if I opened an **empty** box and counted the number of apples in it I would get 0! So 0 can be a counting number. –5 (pronounced 'negative five') is **not** a natural, counting or whole number. You can't count –5 apples.

Negative numbers

Negative numbers are numbers that are smaller than zero. They are a bit hard to understand because you don't usually deal with negative numbers in real life. For instance, it's pretty easy to think about a person carrying around 5 apples, but it's a lot harder or impossible to imagine a person carrying –3 apples around. Negative numbers are very useful though, and not just in maths. Take temperature for instance – if you live in a cold climate, chances are that at night during winter the temperature may drop to below zero – a negative temperature.

Addition using negative numbers

It's a little tricky at first to do calculations with negative numbers. One of the most confusing things is that a negative or positive sign can have two meanings. Look at the following expression:

$$5+4$$

If you read that expression out aloud, you'd say something like "five plus four". In this case, the '+' sign is telling us that we need to add the '4' to the '5'. This is the first type of meaning that a plus or negative sign can have – when they are being used as an *operator*.

Here's another expression:

$$-5+4$$

In this example, we have both a '+' sign and a '–' sign. The '–' sign in this expression is *not* an operator. It is just telling us that the 5 is a negative number – 'negative five'. The '+' sign is still an operator though – it's telling us to 'add four'.

When you do any sort of calculations with negative and positive numbers, it sometimes can help to put the numbers and their signs in brackets. Take this expression for example:

$$5+4$$

We could rewrite this as:

$$(5)+(4)$$

Now, this might seem pointless for such a simple example, but what about something like:

$$-5\times-3+-2$$

There are lots of different signs here, some are operators, and some are just telling us whether a number is negative or positive. To put brackets around the numbers, just follow

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this simple procedure:

Handy Hint #1 - Introducing brackets

Put brackets around the first number and any signs in front of it.

For the other numbers:

If a number has only *one* sign in front of it, put brackets around just the number.

If the number has *two* signs in front of it, put a bracket around both the number and the sign just in front of it.

Let's use this procedure to rewrite $-5\times-3+-2$:

$$\begin{aligned}&\Rightarrow -5\times-3+-2\\&=(-5)\times(-3)+(-2)\end{aligned}$$

So now we know how to rewrite the numbers. We've just got to work out how to do the calculations. To do this, we need a few simple rules:

For addition and subtraction

$$\begin{aligned}'+-'\text{ or }'-+' &= '-'\\ '++' &= '+'\\ '--' &= '+'\end{aligned}$$

For multiplication and division

- Positive '×' or '÷' positive = positive
- Positive '×' or '÷' negative = negative
- Negative '×' or '÷' positive = negative
- Negative '×' or '÷' negative = positive

Here's another way of remembering how this all works which I find easier to remember:

- Any operation except for '–' (+, ×, ÷) with two positive numbers always has a positive answer
- Addition with two negative numbers always gives you a negative answer.
- Multiplying or dividing two negative numbers always gives a positive answer
- Multiplying or dividing a negative and positive number always gives you a negative answer.

So if we go back to our example:

$$(-5)\times(-3)+(-2)$$

So, after we have written our brackets, we realise there are two operations in this expression – a ‘×’ and a ‘+’. Let’s do the ‘×’ operation first:

$$(-5) \times (-3) + (-2)$$

So we’re multiplying two negative numbers together: –5 and –3. We know that this always gives a positive number:

$$= 15 + (-2)$$

Now we have a ‘+–’ situation, which we know is like a ‘–’, so we can write:

$$= 15 - 2$$

$$= 13$$

And that’s how we get our answer: 13. Here’s one more example:

$$-4 \div -2 + -3$$

First we need to write in our brackets. Remember that the first number always has a bracket around it and any sign before it:

$$(-4) \div (-2) + (-3)$$

Once we’ve written our brackets in, we can see that we have two operations: ‘÷’ and ‘+’. We do the division first:

$$(-4) \div (-2) + (-3)$$

We know that any multiplication *or* division with two negative numbers gives you a *positive* answer, so we just need to calculate what 4 divided by 2 is:

$$2 + (-3)$$

We also know that this ‘+–’ combination is the same as ‘–’, so we can rewrite it:

$$=2 - 3$$

$$= -1$$

And there’s our answer, –1.

Minus signs outside brackets

You have to be extra careful when you have a minus sign in front of some brackets like this:

$$-(4 - 2)$$

Now, this is the same as:

$$-1(4 - 2)$$

The –1 is really multiplying everything inside the brackets:

$$-1 \times (4 - 2)$$

When you multiply any number by ‘–1’, the sign of the number changes. For instance, if I multiply ‘–3’ by ‘–1’, it becomes ‘+3’ – it changes from a negative to a positive number. So if I wanted to get rid of the brackets I’d need to multiply each *term* inside the brackets

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by '–1'.

So first I get negative one multiplied by *positive four*:

$$-1 \times 4 = -4$$

I also get negative one multiplied by *negative two*:

$$-1 \times -2 = 2$$

When I put these two bits back together by adding them I get:

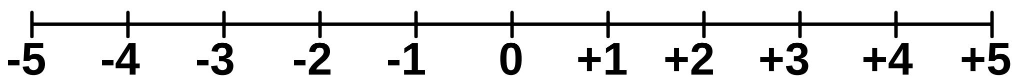
$$-4 + 2$$

Compare this with the original brackets which were: (4 – 2). The positive four became negative four, and the negative two became positive two – both numbers changed their sign. So:

$$-(4 - 2)$$
$$= -4 + 2$$
$$= -2$$

SECTION 1.2 - THE NUMBER LINE

The number line is a diagram you can draw to help you understand and also perform calculations with positive and negative numbers. It's a horizontal line, with marks along it that represent numbers.



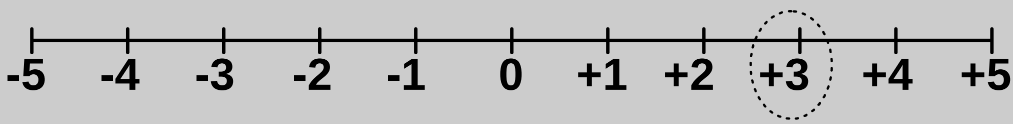
You can do addition and subtraction calculations using the number-line as well. Say I have to find the answer to the following question:

Number line question

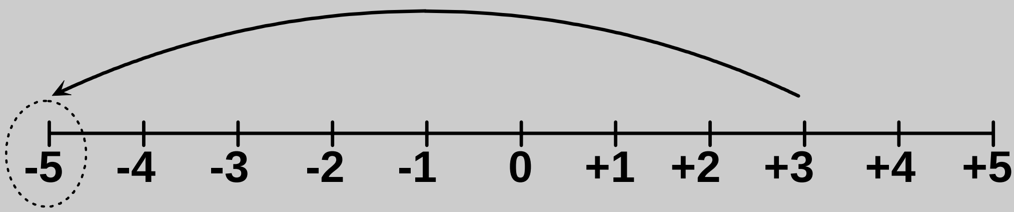
Calculate 3 – 8 – (–4)

Solution

Since this question only has additions and subtractions in it, I can work through it from left to right. So first, I have a 3, which I can find on the number line:



Now I have to *take away* 8. When you take away or subtract using the number line, you move to the *left*. So to take away 8, I need to move 8 steps towards the left:



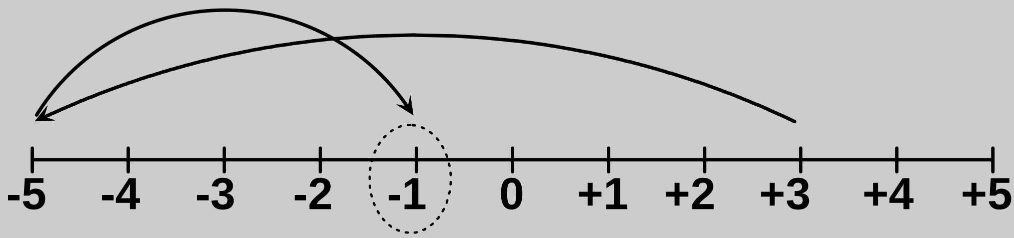
This puts me at *negative five*. The next thing I need to do is *subtract negative four*. We have to be a little bit careful here – we are subtracting a number (–4) that is already negative. When you have:

Something – a negative number

It is the same as:

Something + the positive of that number

In our case, when we subtract '–4' this is the same as adding '+4'. When we do addition on the number line, we move to the right. So, we need to move four places to the right:



Now that we've carried out all the operations in the question, our current position on the number line is the answer: '–1'.

Negative numbers on the calculator

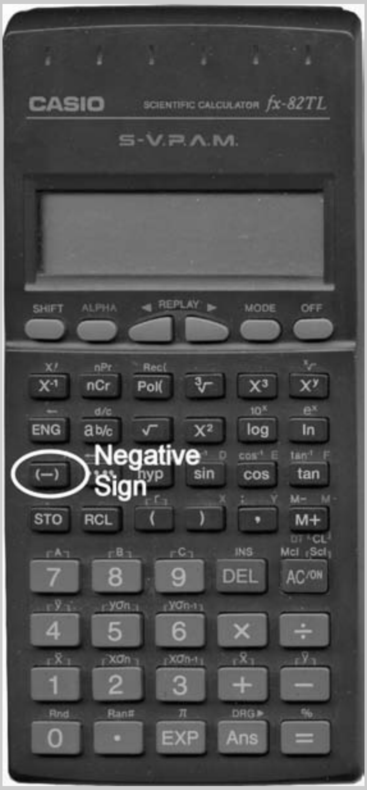
Your calculator is a great way of checking whether what you've done with your positive and negative numbers is correct. Say we had to find an answer to the following question:

$$-5\times 3 - {}^{-}5$$

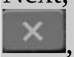
Notice the little negative sign up high in front of the second '5'. This is how people sometimes indicate a number is positive or negative, by putting the sign up high in front of the number.

There are two operations in this question, a '×' and a '–'. Multiplication is performed before subtraction, so it is done first. Here's an explanation of how to solve this problem using two common types of calculators.

Negative numbers on the calculator



First we need to enter in '–5'. To indicate that the number is negative, we first need to press the  button. Then we press the '5' button.

Next, we need to type in the multiply symbol , followed by the number we're multiplying by: .

The next operation is subtraction – so we press the  button.

We are subtracting a *negative* number, so we need to tell the calculator that it is a negative number – we do this by pressing the  button again, before we type in the number: '5'.

First we need to enter in '–5'. First you need to type in '5'. Then we need to tell the calculator that it is *negative* 5 – to do this press the  button. You should see a negative sign appear in front of the '5'. You can press the  button as many times as you want – the number on the screen will switch between being positive and negative. For this question, we want *negative* 5 – so press it until '–5' shows up on the screen.

Now we need to multiply by 3 – do this by pressing the  button followed by the  button.

Now we need to *subtract negative five*. This can be done by first pressing the  button – this tells the calculator that we want to perform a subtraction.

Press the '=' button, and you should get '-10' as your answer.	Next, we need to tell the calculator that the number to subtract is '-5'. Do this by pressing the  button *and then* pressing the  button to tell the calculator it's *negative* five. Finally, to get the answer, press the '=' button.

Composite numbers

Composite numbers are numbers that you can get by multiplying together two or more natural numbers which are not 1. The two or more numbers that you multiply can be the same or different. For instance, 8 is a composite number because you can get it by multiplying 4 by 2, or even by multiplying 2 by 2 by 2. 15 is a composite number because you can get it by multiplying 5 by 3. 7 **is not** a composite number because the only way you can get 7 is by multiplying 1 and 7. Remember, to be a composite number, none of the numbers you multiply together to get it can be 1.

Factors and products

Since we're talking about numbers that multiply to give other numbers, we should learn what factors and products are. The *factors* of a number are the natural numbers that multiply together to make that number. Take for instance the number 15. We can get 15 in two different ways:

$$\text{product} \quad \left\{ \begin{matrix} 15 = 15 \times 1 \\ 15 = 5 \times 3 \end{matrix} \right\} \quad \text{factors}$$

Products are the numbers that you get by multiplying other numbers together. So in the maths above, the number 15 is what we're getting as a result of doing multiplication – 15 is the product. The numbers on the right of the equals sign are the ones we are multiplying together to give 15, so they are the factors. So the factors of 15 are 15, 1, 5, and 3 (yes, 15 is a factor).

Prime numbers

Prime numbers are natural numbers which you can only get by multiplying 1 and themselves. For instance, 5 is a prime number because you can only get 5 by multiplying 1 and 5 together. 4 **is not** a prime number because you can get 4 by multiplying 2 and 2 together. Prime numbers have exactly two unique factors – themselves and 1. For instance, take the prime number 7. The only factors (numbers that multiply together to give it) are 1 and 7. Composite numbers on the other hand, have more than two factors.

Is 1 a prime number?

1 is a special type of number, and is not considered a prime number. This sort of makes sense. What natural numbers can you multiply together to get 1? Well, we know that 1 by 1 gives you 1. So the only factor of 1 is 1. Since 1 has only one factor, it isn't a prime

number – prime numbers have exactly two unique factors.

Square numbers

Square numbers are numbers that you can get by multiplying a natural number by itself. For instance, if I start with the natural number 3, and multiply it by itself:

$$=> 3 \times 3$$
$$= 9$$

I get 9, which is a *square* number. People sometimes call square numbers *perfect squares*. Here's a list of the first few square numbers:

Number	Square	Number	Square
1	1	11	121
2	4	12	144
3	9	13	169
4	16	14	196
5	25	15	225
6	36	16	256
7	49	17	289
8	64	18	324
9	81	19	361
10	100	20	400

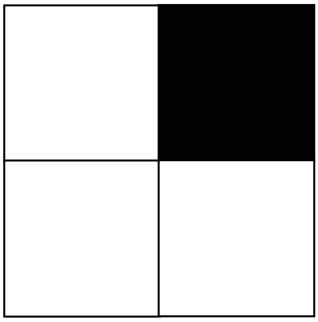
Integers

An *integer* is another type of number. Integers are similar to natural, counting and whole numbers except for one thing. –3 is an *integer*. So is –10. So is –55. 0 is also an integer. So integers are like natural numbers but they can be negative as well.

SECTION 1.3 - FRACTIONS

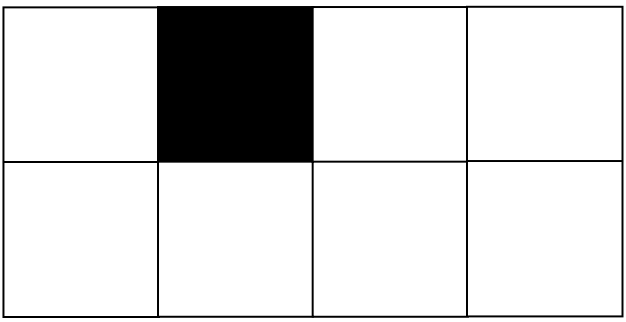
Let's forget about mathematics for a moment. If I say something like "I only have a fraction of an apple left" what do I mean? I mean that I only have part of the entire apple left, probably because I have already eaten some of it. I could also say something like "I only have half an apple left".

So a fraction of something means only a certain part of the entire thing. Look at the shape below:



What *fraction* of this shape is dark? If you look at the shape, you can see it is made up of four squares. **One out of four** of these squares is dark. So I could say something like, "One out of four of these squares is dark." But I want to know what *fraction* of the shape is dark. So instead, I would say something like, "**One quarter** of the shape is dark." I could also say, "**One fourth** of the shape is dark."

Let's try another example:



In this picture, what *fraction* of the shape is dark? Well, looking at the picture, I can see that the shape is made up of eight squares. And only one of these squares is dark. So one out of eight squares is dark. To answer the question, "**One eighth** of the shape is dark."

Writing fractions

So now we know a little about fractions. How do we write fractions in proper mathematical *notation*? Let's take a simple fraction – "one fourth". Well, first let's change how we say that fraction. Instead of "one fourth" let's say, "one out of four." Now, to write the fraction, write down the first number you just said:

$$1$$

Now draw a short horizontal line underneath the number you just wrote:

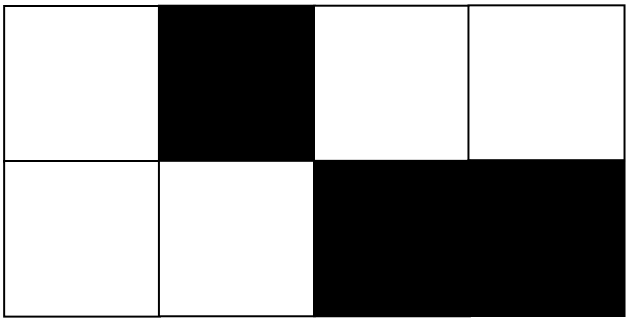
$$\underline{1}$$

Now write the *second* number you just said underneath this line:

$$\frac{1}{4}$$

And there you go! You've just written a fraction.

Okay, let's try a slightly more complicated question. Write down the fraction of the shape below that is dark:



Well, first we look at the shape and work out that 3 out of 8 of the squares are dark. So, I take the first number and write that:

$$3$$

Then I add a horizontal line underneath this number:

$$\frac{3}{\ }$$

The last thing to do is write down the second number, 8:

$$\frac{3}{8}$$

And I'm finished. You may want to use the fraction you've just written in a complete sentence:

"$\frac{3}{8}$ of the shape is dark"

Fractions can be written in lots of different ways. Below are some other ways you can write this same fraction.

With a slanting line	3/8
With a different slanting line	³⁄₈

How some calculators display fractions	3⌟8

One thing that these different ways of writing a fraction have in common is that they involve *two* numbers. The *general form* of a fraction is:

$$\frac{\text{Numerator}}{\text{Denominator}}$$

So for 3/8, the *numerator* is 3 and the *denominator* is 8. Often the numerator and denominator are described as the "top and bottom" of a fraction.

Proper fractions

Proper fractions are fractions where the numerator is *smaller than* the denominator. The following fractions are proper fractions:

$$\frac{1}{3}, \frac{5}{27}, \frac{7}{57}, \frac{3523}{764442}$$

Equivalent fractions

The same fraction can be written using different numbers. For instance, let's try working with a very simple fraction, one half:

$$\frac{1}{2}$$

Now, with proper fractions, you are allowed to multiply *or* divide the numerators and denominators, as long as you do the same thing to both. For example, we could multiply both the numerator and the denominator by 10 and we would have an *equivalent* fraction:

$$\Rightarrow \frac{1}{2} \times \frac{10}{10}$$
$$= \frac{10}{20}$$

Remember, we can also divide too. For instance we could divide top and bottom by 5:

$$\Rightarrow \frac{10}{20} \div \frac{5}{5}$$
$$= \frac{2}{4}$$

So 2/4 is another fraction that is *equivalent* to 1/2.

Improper fractions

Improper fractions have denominators that are *larger or equal to* the numerator. Improper fractions are numbers that are *larger or equal* to one. The following fractions are improper fractions:

$$\frac{17}{5}, \frac{4}{2}, \frac{37}{22}, \frac{645}{276}$$

Mixed numbers

Improper fractions can also be written as *mixed* numbers. Say we have the improper fraction 7/5. We could also describe this fraction as "seven fifths". Now, we know that "five fifths" make one. So let's split up our fraction like this:

$$\begin{aligned}
&\Rightarrow \frac{7}{5} && \text{Here's the original improper fraction} \\
&= \frac{5+2}{5} && \text{Let's split up the numerator} \\
&= \frac{5}{5} + \frac{2}{5} && \text{Let's split it up into two separate fractions} \\
&= 1 + \frac{2}{5} && \frac{5}{5} \text{ is the same as } 1 \\
&= 1\frac{2}{5} && \text{There is no need to write the plus sign}
\end{aligned}$$

The last term shown above is known as a mixed number, simply because it *mixes* both whole numbers *and* fractions. Mixed numbers can also be converted back into improper fractions. Say we start with this mixed number:

$$\begin{aligned}
&\Rightarrow 2\frac{5}{7} && \text{Here's the original mixed number} \\
&= 2 + \frac{5}{7} && \text{Let's put the plus sign back in} \\
&= \frac{14}{7} + \frac{5}{7} && \text{Now 2 is the same as 14/7} \\
&= \frac{14+5}{7} && \text{Since 14 and 5 are over the same denominator we can put them together...} \\
&= \frac{19}{7} && \text{to get an improper fraction.}
\end{aligned}$$

Reciprocals

The *reciprocal* of a fraction is like a fraction's partner. To find the reciprocal of a fraction, all you have to do is swap the numerator and the denominator. Let's find the reciprocal of a simple fraction, 7/9:

$$\text{Find the reciprocal of } \frac{7}{9}$$

$$\text{Swap the numerator and the denominator}$$

$$\frac{9}{7} \text{ is the reciprocal of } \frac{7}{9}$$

Reciprocals have an interesting *property*. When you multiply a fraction by its reciprocal you get 1. We can do this with the fraction we were just working with:

$$\Rightarrow \frac{7}{9}\times\frac{9}{7}$$
$$=\frac{7\times 9}{9\times 7}$$
$$=\frac{63}{63}$$
$$=1$$

You can also find the reciprocals of whole numbers. For instance, to find the reciprocal of 5, you have to remember that:

$$\Rightarrow 5$$
$$=\frac{5}{1}$$

Now swap the numerator and denominator

$$\frac{1}{5}\text{ is the reciprocal of }\frac{5}{1}$$

Simplifying fractions

Fractions are not always given in their simplest form. For instance the fraction $^{868}/_{1116}$ can be simplified all the way down to $^{7}/_{9}$. To simplify a fraction, you need to find the *greatest common factor* (GCF) for the numerator and denominator.

So say we had a fraction $^{63}/_{98}$. To simplify this fraction we first have to find the GCF of 63 and 98:

- 63 has factors 1, 3, 7, 9, 21
- 98 has factors 1, 2, 7
- The largest common factor is 7.
- So the GCF of 63 and 98 is 7.

Now what do we do with this GCF? Well, we use it to divide both the numerator and denominator of the fraction:

$$\Rightarrow \frac{63}{98}\div\frac{7}{7}$$
$$=\frac{9}{14}$$

$^{9}/_{14}$ is the *simplest* form of $^{63}/_{98}$.

Greatest common factor

First of all, this is often *abbreviated* to GCF standing for (G)reatest (C)ommon (F)actor. The GCF of two numbers is the largest whole number that evenly divides both numbers. This sounds confusing, so we'll do an example to show how it all works.

Greatest common factor question

Find the greatest common factor of 24 and 48.

Solution

We need to find the biggest number that evenly divides both 24 and 48. When you *evenly divide* something, you aren't left with a fraction or remainder. For example, 2 evenly divides 8 since 8 ÷ 2 = 4 exactly, with no remainder. So let's see if we can find the GCF of 24 and 48 by starting with 1:

- 1 evenly divides 24 since 24 / 1 = 24 exactly.
- 1 evenly divides 48 since 48 / 1 = 48 exactly.

Let's try a bigger number:

- 2 evenly divides 24 since 24 / 2 = 12 exactly.
- 2 evenly divides 48 since 48 / 2 = 24 exactly.

How about 3:

- 3 evenly divides 24 since 24 / 3 = 8 exactly.
- 3 evenly divides 48 since 48 / 3 = 16 exactly.

Let's try 4:

- 4 evenly divides 24 since 24 / 4 = 6 exactly.
- 4 evenly divides 48 since 48 / 4 = 12 exactly.

Hmmm…let's try 5 then:

- 5 *does not evenly divide* 24 since 24 / 5 = $4\,^{4}/_{5}$
- 5 also *does not evenly divide* 48 since 48 / 5 = $9\,^{3}/_{5}$

So what are we supposed to do now? Someone who has never done this before might be tempted to say that 4 is the GCF of 24 and 48 since it was the last number we tried that divided evenly into 24 and 48. However, let's keep going and see where we get, starting with 6:

- 6 evenly divides 24 since 24 / 6 = 4 exactly.
- 6 evenly divides 48 since 48 / 6 = 8 exactly.

So, although 5 didn't divide evenly into the two numbers, we find that 6 does. So, how do you know when to stop trying larger numbers? The simple answer is you don't! To be safe, you need to keep trying numbers until you hit the size of the smallest number – in this case 24. Sometimes the GCF is called the (H)ighest (C)ommon (F)actor – HCF.

Greatest common factor – one technique

Here's one step by step procedure for finding the greatest common factor.

Handy Hint #2 - Greatest common factor

Here's one technique for finding the greatest common factor which I have found the easiest to use. It's a 4 step process:

Step 1: List all the factors of the first number

Step 2: List all the factors of the second number

Step 3: Find all the numbers that *are common to* (in other words, are in) both lists.

Step 4: The largest number found in step 3 is the GCF.

Let's try this with 24 and 48:

Step 1: Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

Step 2: Factors of 48: 1, 2, 3, 4, 6, 8, 12, 24, 48

Step 3: Common factors: 1, 2, 3, 4, 6, 8, 12, 24

Step 4: The largest number in step 3 is 24

So the GCF of 24 and 48 is 24. Let's try a harder example:

Greatest common factor question

Find the GCF of 24 and 132

Solution

Step 1: Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

Step 2: Factors of 132: 1, 2, 3, 4, 6, 11, 12

Step 3: Common factors: 1, 2, 3, 4, 6, 12

Step 4: The largest number in step 3 is 12

So the GCF of 24 and 132 is 12.

There are also a few different ways you can talk about fractions. The table below shows ways you can say each fraction:

Fraction in number form	Word form 1	Word form 2	Percentage form	Decimal form
1/2	One half	One on two	50 percent	0.5
1/3	One third	One on three	33 percent	0.33
1/4	One quarter	One on four	25 percent	0.25
1/5	One fifth	One on five	20 percent	0.20
1/6	One sixth	One on six	17 percent	0.17
1/7	One seventh	One on seven	14 percent	0.14
1/8	One eighth	One on eight	13 percent	0.13
1/9	One ninth	One on nine	11 percent	0.11
1/10	One tenth	One on ten	10 percent	0.10

Fractions as percentages

You can also write fractions as percentages. For instance, instead of saying "Half of" or '$^1\!/_2$' of you can also say "50% of". To work out what percentage a fraction is *equivalent* to is quite easy. Let's try converting 3/8 into a percentage.

First of all, we know that 1 is the same as 100 %. But in this case, we are not dealing with ones, we are dealing with eighths. So take that 100%, and divide it by 8. You may want to use your calculator to do this:

$$100\% \div 8 = 12.5\%$$

Okay, so we have split the 100% up into eight bits, and each bit is 12.5%. Now, let's look at the fraction we're dealing with, 3/8. Remember that:

$$\frac{3}{8} = 3 \times \frac{1}{8}$$

This is just another way of saying the fraction – we have 3 lots of $1/8^{th}$. Now, we just calculated that 1/8 is 12.5%. So if we have 3 of them, then we have:

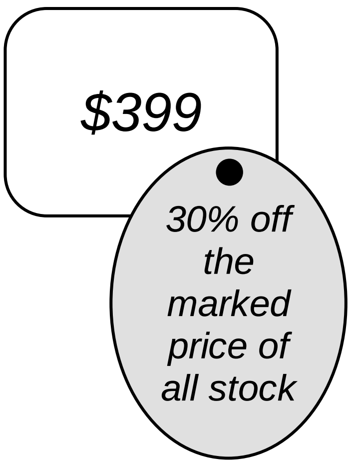
$$\Rightarrow 3\times\frac{1}{8}$$
$$= 3\times 12.5\%$$
$$= 37.5\%$$

And there we have our answer. 3/8 is the same as 37.5%.

SECTION 1.4 - PROBLEMS USING PERCENTAGES

In real life you often use or hear about percentages. For example, if you're at the shops you might see a stereo on sale. On the sign you might see something like:

In another store, you might see a sign for that same stereo:



So in both stores, the stereo's normal price is $399.00. However, in one store, this has been reduced to $249.00. In the *other* store, the stereo's original price of $399 has been reduced by **30%,** *but no-one's written what the new price is*. How do you know which store is cheaper? Well you could do it three ways. You could ask the store assistant to work out what 30% off $399.00 is, you could work it out yourself, or you could work out what percentage discount the $249 price worked out to. Let's try the third option.

So we have an original price, and we have a sale price. To get percentages, you usually divide one number by another, and then turn it into a percentage. To work out what the percentage discount was, we first need to work out what percentage $249 is of $399. We can do this by dividing $249 by $399 and turning it into a percentage, like this:

$$=> \frac{\$249}{\$399}$$
$$= 0.6241$$

To convert to a percentage, we need to multiply by 100:

$$\% = 0.6241 \times 100$$
$$\% = 62.41\%$$

So $249 is about 62% of $399. However, the percentage discount indicates how much percent of the original price has been taken off. Since we're left with only 62% of the original price, this must mean 38% of the original price was taken off. In mathematical terms:

$$\% \text{ discount} = 100\% - \text{percentage of original price left}$$

So for this problem:

$$\% \text{ discount} = 100\% - \text{percentage of original price left}$$
$$\% \text{ discount} = 100\% - 62.41\%$$
$$\% \text{ discount} = 37.59\%$$

So the discount at the first store is about 38%, which is a larger discount than at the second store.

Don't be a sucker at clearance sales

Recently I went to one of those clearance sales in a large building where they sell a whole heap of clothing etc… at cheap prices. Now, this particular sale had been advertised all week on the television, with the people on the advertisement shouting out repeatedly, "50% off the original marked price." So I figured, half-price clothes – that's a pretty good deal, right? So I went along and had a look, which was when it got interesting…

While I was there, which also happened to be the last day of the sale, an announcement was made over the speaker system. "Final day price cut – take fifty percent off the original price, and then a further forty percent off that". I heard people all around me going crazy as they did the maths aloud: "Well, that's fifty percent off the price plus another forty percent off the price…that must mean it's ninety percent off the original price". At that point, everyone went nuts and bought as much as they could carry. However, when everyone got to the cash registers, the cost was much more than they had thought, and so they had to return a lot of the stuff. So what went wrong?

Well, these people didn't listen carefully enough to the announcement, especially the last bit: "Final day price cut – take fifty percent off the original price, *and then a further forty percent off that.*" Let's do this with a pretend item which originally cost $100 (this way $1 corresponds to 1%).

So I start with the $100 price. Then I take off 50% off this. 50% of $100 is $50, so I'm left with $50. Everything seems fine so far.

Now, the second part of the announcement – "…and then a further forty percent off that." "That" refers to the $50 price I'm left with. So this means I take 40% off the $50. If I'm taking off 40%, that means I'm left with 60% of $50:

$$\begin{aligned} &=> 60\% \times \$50 \\ &= 0.6 \times \$50 \\ &= \$30 \end{aligned}$$

So my final sale price is $30. If you compare this with the original sale price, you'll see that $30 is 30% of $100. So the overall percentage discount was:

$$\begin{aligned} \%\ \text{discount} &= 100\% - \text{percentage of original price left} \\ \%\ \text{discount} &= 100\% - 30\% \\ \%\ \text{discount} &= 70\% \end{aligned}$$

So why did everyone think that the overall discount was 90%? Well, the problem came from the "*…and then a further forty percent off that*" part of the announcement. The extra 40% discount **was not** an extra 40% off the original price. It was a further 40% discount on the *already discounted* price. Only the 50% off part applied to the *original price*.

This shows how important it is, both in real life, and in school, to make sure you read or listen very carefully, and think about exactly what is meant. In this case, failure to notice a couple of key words – "*and then….off **that***" meant a whole lot of people had a very wrong idea of how much things cost.

Commission

If you ever work as a salesperson, there is a good chance that you will have the opportunity to earn *commission* on anything you sell. Commission is like an incentive for a salesperson to make lots of sales, because for each sale they make they get paid a small fraction of the sale price. Even though the fraction or percentage may be very small, it can make a big difference to the amount of money you earn over an entire week. Commission becomes *very important* when you're selling expensive things, like cars, computers or houses. Even a small percentage of $300,000 is a lot of money!

So if you're ever offered a job as a salesperson, check out what the salary is like, and whether you get commission. It's also important to find out what items the commission applies to – sometimes you only earn commission on certain brands. Also, it's very important to know what the percentage commission is.

So say you've just been offered a new job at an electronics store. You get a basic salary of $400 a week (not great for a full time job), but you can also earn commission on any laptop computers you sell. The commission is 2% of the price of the laptop. The manager tells you that the last salesperson managed to sell about 3 laptops a week. You want to know how much money you'd pull in a week based on what he told you. First thing to do is write down a statement saying how much money you could earn a week:

Earnings per week = $400 + Commission

Next thing to do is write down what that commission actually is:

Earnings per week = $400 + 2% on laptop sales

Let's put in how many laptops we might be selling a week as well:

Earnings per week = $400 + 2% on sales of 3 laptops

So 2% on laptop sales hey? Well this ain't much use unless you know how much the laptops cost. This is where you make an *educated* guess. I'm gonna guess that a typical laptop costs $2500. So let's put that in:

Earnings per week = $400 + 2% on sales of 3 laptops at $2500 a laptop

Now we just need to make it into a mathematical statement and turn some of the words into mathematical symbols and operators. "2% of" becomes a multiplication symbol. The "3 laptops at $2500 a laptop" also becomes a multiplication:

$$\text{Earnings per week} = \$400 + 2\% \times 3 \times \$2500$$

2% of something is like multiplying it by $\frac{2}{100}$ or 0.02. So it becomes:

$$\text{Earnings per week} = \$400 + 0.02 \times 3 \times \$2500$$
$$\text{Earnings per week} = \$400 + \$150$$
$$\text{Earnings per week} = \$550$$

If you do something like this in real life, you've got to remember that the only guaranteed money you'll earn each week is $400. Earning more than this will depend on whether you manage to earn any commission – in this case, this translates to whether you sell any laptops.

SECTION 1.5 - DOING CALCULATIONS WITH FRACTIONS

You can do all the normal calculations with fractions – addition, subtraction, multiplication and division. There are however some special tricks that are good to know.

Addition with fractions

To add together two fractions, you have to make sure that they have a *common* denominator. In other words, they must both have the same number on the bottom. This won't always be the case, so sometimes you have to do some *manipulation* of the fractions before you can add them together.

Say we have the following problem:

Fraction addition problem

Add together $\frac{4}{7}$ and $\frac{17}{21}$

Solution

Notice that the fractions do not have the same denominator. Before we can add them together, we will have to make sure they do. So how do we go about doing that?

Well, we need to find the *lowest common multiple* (LCM) of 7 and 21. The LCM is the lowest number that you can get by multiplying both numbers by something. We'll find the LCM for 7 and 21 to show you what to do:

First take one of the numbers – we'll start with 7. List all the multiples of 7 like this:

7, 14, 21, 28, 35, 42, 49, 56, 63, 70 …

Now take the other number and do the same thing:

21, 42, 63, 84, 105

Now find the smallest number that is in both lists.

In this case it's very easy – 21 is the smallest number in both lists. 21 is therefore the lowest common multiple of 7 and 21.

Now back to the original fractions. Rewrite both fractions with 21 as the denominator. For $\frac{4}{7}$ to change to $\frac{\textit{something}}{21}$, we need to multiple the top and bottom by 3. This is because 7 multiplied by 3 gives 21, which is what we're trying to get on the bottom. So:

$$\frac{4}{7} \times \frac{3}{3} = \frac{12}{21}$$

And now we have the first fraction in the form we want – with a denominator of 21. Now let's look at the second fraction – in this case we don't have to do anything, it's already got a denominator of 21. So, let's rewrite the original sum and calculate it:

$$\Rightarrow \frac{4}{7} + \frac{17}{21}$$
$$= \frac{12}{21} + \frac{17}{21}$$
$$= \frac{12+17}{21}$$
$$= \frac{29}{21}$$
$$= 1\frac{8}{21}$$

So addition isn't very hard at all. Subtraction is quite similar to addition as you'll see in the next section.

Subtraction with fractions

When you subtract a fraction from another one, you follow the same process as for addition. Let's do an example:

Fraction subtraction question

Solve $\frac{3}{9}-\frac{2}{7}$

Solution

First of all, we need to find the LCM of the denominators. So let's list the multiples of each number, starting with 9:

9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99

and also for 7:

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98

Now we have to find the smallest number that is in both lists. After searching for a bit you will find that 63 is the smallest number common to both lists. So 63 is the LCM of 9 and 7.

So now we have to convert our original fractions so that they both have a denominator of 63. For $\frac{3}{9}$ we know that we have to multiply the 9 by 7 to get 63, so:

$$\frac{3}{9}\times\frac{7}{7}=\frac{21}{63}$$

And we do the same thing for $\frac{2}{7}$. We can work out that to get 63 on the bottom, we have to multiply 7 by the number 9:

$$\frac{2}{7}\times\frac{9}{9}=\frac{18}{63}$$

Now we can do the calculation:

$$\Rightarrow \frac{3}{9} - \frac{2}{7}$$
$$= \frac{21}{63} - \frac{18}{63}$$
$$= \frac{21-18}{63}$$
$$= \frac{3}{63}$$
$$= \frac{1}{21}$$

Now, this is the correct answer to the problem. However, we haven't done it in the easiest way. If you look back at the original question, you should notice that one of the fractions isn't in its simplest form; $\frac{3}{9}$ simplifies down to $\frac{1}{3}$! So we should have done this simplification first before we did the rest of the process. This is something to look for in all problems – make sure you have made the question as simple as possible before you carry out the calculations.

Multiplication with fractions

Multiplying fractions is in many ways easier than doing addition or subtraction. All you have to do is follow a simple three step process:

Handy Hint #3 - Multiplying with fractions

Multiply the tops of the fraction, in other words the numerators.

Multiply the bottoms of the fractions, in other words the denominators.

Simplify the fraction you get from the multiplication.

So, let's do an example:

Fraction multiplication question

Solve $\frac{4}{7} \times \frac{9}{5}$

Solution

Now, step 1 is to multiply the tops of the fractions together:

$$\frac{4\times 9}{7\times 5}=\frac{36}{7\times 5}$$

Step 2 is to multiply the bottoms of the fractions together:

$$\frac{36}{7\times 5}=\frac{36}{35}$$

Now we have a single fraction, which is our answer. However, there is still step 3 to consider – we have to write the fraction in its simplest form. The current fraction is an *improper* fraction – the numerator (top) is bigger than the denominator (bottom). We can change it into a *proper* fraction by splitting it up into a whole number and a fraction:

$$\Rightarrow \frac{36}{35}$$
$$=\frac{35+1}{35}$$
$$=\frac{35}{35}+\frac{1}{35}$$
$$=1\frac{1}{35}$$

Dividing with fractions

Dividing fractions is only a little bit harder than multiplying. There is only one extra little step, and it's an easy one. Here are the steps for dividing a fraction:

Handy Hint #4 - Dividing with fractions

Swap the numerator (top) and denominator (bottom) of the fraction doing the dividing.

Multiply the tops of the fraction, in other words the numerators.

Multiply the bottoms of the fractions, in other words the denominators.

Simplify the fraction you get from the multiplication.

The extra step is in bold (the thicker letters). Let's do an example:

Fraction division question

Solve $\frac{12}{5} \div \frac{3}{2}$

Solution

Step 1 is to swap the top and bottom of one of the fractions (remember only do it to one of them):

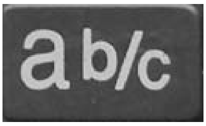
$$\Rightarrow \frac{12}{5} \div \frac{3}{2}$$
$$= \frac{12}{5} \times \frac{2}{3}$$

Note how the 3 and 2 have swapped places. Also look at how we are now multiplying the fractions in the second line, instead of dividing them. Now we just do a normal multiplication:

$$=> \frac{12}{5} \times \frac{2}{3}$$
$$= \frac{12 \times 2}{5 \times 3}$$
$$= \frac{24}{15}$$
$$= 1\frac{3}{5}$$

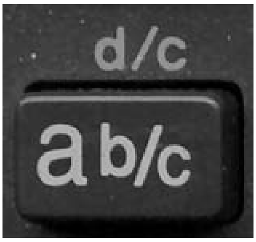
Fractions on your calculator

You can use a calculator to make working with fractions a lot easier. What you need to do is find the fraction button on your calculator. There should be two symbols somewhere on your calculator to do with fractions: one which looks like:

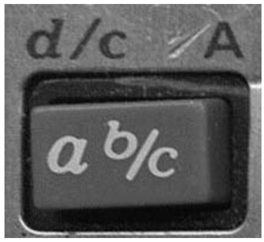


and one that looks like:

Usually, the first symbol is on a button somewhere on your calculator, and the second symbol is above this button, like this:



or



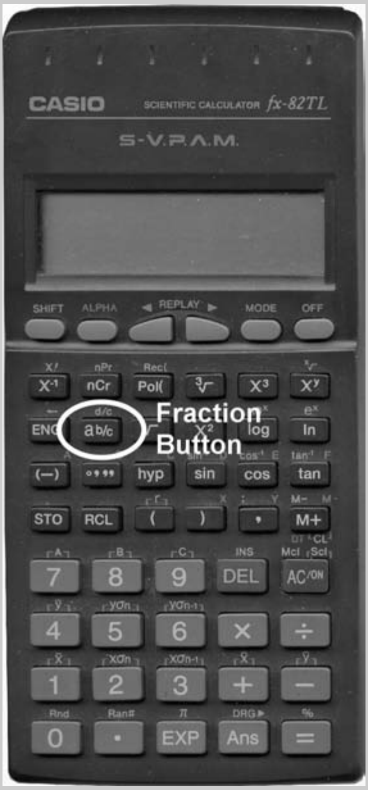
To show how to use a calculator to do fractions, we'll do a few example questions:

Calculator fractions question

- Write $38/5$ as a proper fraction.
- Write $16/17$ as a percentage.
- Add $15/7$ and $18/53$.
- Calculate the following: $\frac{37}{23} \div \frac{14}{3} - \frac{16}{7}$ and give the answer as an improper fraction.

Solution

Fractions on your calculator



a)

Type in 38.

Press the fraction button (a b/c).

Type in 5.

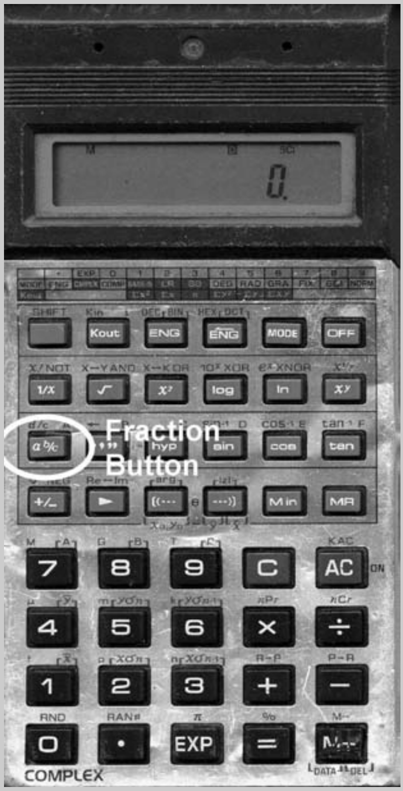
Press the equals sign and you should get 7⌟3⌟5. This should be read as $7\frac{3}{5}$.

b)

Type in 16.

Press the '÷' button.

Type in 17.



a)

Type in 38.

Press the fraction button (a b/c).

Type in 5.

Press the equals sign and you should get 7⌟3⌟5. This should be read as $7\frac{3}{5}$.

b)

Type in 16.

Press the '÷' button.

Type in 17.

Press ‘=’.

Press the ‘×’ button.

Type in 100.

Press ‘=’. You should have 94.1176… This is your answer, as a percentage. So, 94.12 %.

c)

Type in 15.

Press the fraction button.

Type in 7.

Press the ‘+’ button.

Type in 18.

Press the fraction button.

Type in 53.

Press the ‘=’ button.

You should have 2⌟179⌟371. This should be read as $2\frac{179}{371}$

d)

Type in 37.

Press the fraction button.

Type in 23.

Press the ‘÷’ button.

Type in 14.

Press the fraction button.

Type in 3.

Press ‘=’.

Press the ‘×’ button.

Type in 100.

Press ‘=’. You should have 94.1176… This is your answer, as a percentage. So, 94.12 %.

c)

Type in 15.

Press the fraction button.

Type in 7.

Press the ‘+’ button.

Type in 18.

Press the fraction button.

Type in 53.

Press the ‘=’ button.

You should have 2⌟179⌟371. This should be read as $2\frac{179}{371}$

d)

Type in 37.

Press the fraction button.

Type in 23.

Press the ‘÷’ button.

Type in 14.

Press the fraction button.

Type in 3.

Press the '=' button.

You should now have $\frac{111}{322}$, we're halfway there.

Press the '–' button.

Type in 16.

Press the fraction button.

Type in 7.

Press the '=' button. You should now have -1⌟303⌟322. This should be read as $-1\frac{303}{322}$. Now we have to change it into an improper fraction. For this we need the [d/c] function. Since this is a *secondary* function on most calculators, you need to press the '2nd function' or shift key first, and then press the [a b/c] button. So...

With -1⌟303⌟322 on your display, press the [SHIFT] button.

Now press the [a b/c] (because you pressed the shift button first, you're really using the [d/c] function now).

Your display should now read:

–625⌟322, which is the answer as an improper fraction. If you were to write this fraction on paper, you'd write:

$$-\frac{625}{322}$$

Press the '=' button.

You should now have $\frac{111}{322}$, we're halfway there.

Press the '–' button.

Type in 16.

Press the fraction button.

Type in 7.

Press the '=' button. You should now have -1⌟303⌟322. This should be read as $-1\frac{303}{322}$. Now we have to change it into an improper fraction. For this we need the [d/c] function. Since this is a *secondary* function on most calculators, you need to press the '2nd function' or shift key first, and then press the [a b/c] button. So...

With -1⌟303⌟322 on your display, press the [SHIFT] button.

Now press the [a b/c] (because you pressed the shift button first, you're really using the [d/c] function now).

Your display should now read:

–625⌟322, which is the answer as an improper fraction. If you were to write this fraction on paper, you'd write:

$$-\frac{625}{322}$$

SECTION 1.6 - DECIMAL NUMBERS

Decimal numbers are another way of writing fractions or numbers that aren't whole numbers. All decimal numbers have a decimal point in them which separates the part of the number which is larger than one from the part that is smaller than one. Let's look at the following decimal number:

$$215.678$$

We can split this number up into lots of decimal numbers:

$$215.678 = 200.0 + 10.0 + 5.0 + 0.6 + 0.07 + 0.08$$

In word form this is:

2 hundreds + 1 ten + 5 units + 6 tenths + 7 hundredths + 8 thousandths

This table shows all the words for these different *orders of magnitude*:

Number	**Word form**
1 000 000	Millions
100 000	Hundreds of thousands
10 000	Tens of thousands
1 000	Thousands
100	Hundreds
10	Tens
1	Units or ones
0.1	Tenths
0.01	Hundredths
0.001	Thousandths
0.0001	Ten-thousandths
0.00001	Hundred-thousandths
0.000001	Millionths

SECTION 1.7 - ROUNDING

When you use a calculator to perform calculations, you will often get lots and lots of decimal places. In exams or on assignments your teachers probably don't expect you to write down the complete number. There are two ways you can *round* off a number so that you don't have to write all those digits after the decimal point.

Rounding to a number of decimal places

This is the easiest type of rounding to do. Say your teacher told you to find the answer to the following question and give the answer to 3 decimal places:

$$7 \div 9 = ?$$

Well, you can calculate this using your calculator. You should get something like:

$$0.777777777$$

showing up on your calculator screen. The teacher has asked for the answer to be given to *3 decimal places*. To write the answer with the correct number of decimal places, follow this simple procedure:

Handy Hint #5 - Rounding to decimal places

Write the number down, but only to one less than the asked for number of decimal places. So if you've been asked for 3 decimal places, only write down two decimal places.

Now find the 'special' digit. The 'special' digit is the digit located one more decimal place to the right than the number of decimal places you've been asked for. So say you've been asked for 3 decimal places, look at the value of the digit in the *4th* decimal place – this is the 'special' digit.

If the 'special' digit is 0, 1, 2, 3, or 4, then just write your last decimal place in as it is.

If the 'special' digit is a 5 or larger, you need to add one to your last decimal place.

When you're checking the value of the 'special' digit – you're basically checking whether it is larger than 4 or not. If it's *not* larger than 4, then you can write your last decimal place in without changing it. But if the 'special' digit is larger than 4, you need to increase the value of your last decimal place by one.

So back to our example, we've been asked for 3 decimal places, so first we write the number with only 2 decimal places:

$$0.77$$

Now, we're not quite finished yet. We still have to write the 3rd decimal place down in our answer. Before we do this, look at the value of the 'special' digit – the *fourth* digit to the right of the decimal place. If this digit is a 0, 1, 2, 3 or 4, then we just go ahead and write the 3rd decimal place. *However*, in this case, the fourth digit is a '7', so what we have to do is increase the value of the 3rd decimal place by one. So our answer becomes:

0.778

Whenever you round a number to a certain number of decimal places, you need to check to see whether the last digit will need to be rounded up. So if you're rounding to 3 decimal places, you check the fourth digit (the 'special' digit) to the right of the decimal place, and if this digit is larger than '4', you need to round up your last decimal place.

Here's another rounding question:

Rounding question

Calculate 12 ÷ 7 and give the answer to 4 decimal places.

Solution

Well, we can type this into our calculator. You should get something like:

1.7142857

showing up on your screen. Since we've been asked to give our answer to 4 decimal places, we rewrite our answer, but only writing the first three digits to the right of the decimal point:

1.714

One more thing to do – we need to write down the fourth decimal place. Before we do this, we need to look at the value of the 'special' digit – the *5th* digit to the right of the decimal point. In this case, the number is an '8', so we need to round up our fourth decimal place.

So, instead of writing 1.7142, we round up the fourth decimal place to get:

1.7143

Handy Hint #6 - Tricky rounding

Say we're given the number 7.896, and told to round it to 2 decimal places. First, we rewrite the number with only 1 digit to the right of the decimal point:

7.8

Now we need to write down the 2nd digit to the right of the decimal point. In our original number, this is a '9'. But before we write that down, we need to check whether it has to be rounded up or not. To do this, we check the value of the 3rd digit to the right of the decimal place (the 'special' digit) – in this case the digit is a '6'. Since 6 is larger than 4, we need to round up the 2nd digit. But the 2nd digit is a

9 already, which means the 2nd digit becomes a zero and we round up the digit *to the left of it*. Here is the whole process:

Our original answer is:

7.896

First we write it with only 1 decimal place:

7.8

Then we check the value of the 3rd decimal place, the 'special' digit:

3rd digit is a '6'

So have to round up the 2nd decimal place, but it's already a 9.

7.89

So we change the 2nd decimal place to a zero, and round up the decimal place one to the left of it. In this case that means changing the '8' to a '9':

7.90

And there we have our answer, 7.90. This is an example of when the rounding process is a bit tricky. To cope with situations like this however is easy, you just need to remember how rounding works.

If a digit is a '9', and you increase the value of it by 1, then that digit becomes 0, and you increase the value of the digit just to its left by 1.

Here's an even trickier case:

Round 9.99996 to 1 decimal place

So first we write the number with 0 decimal places:

9

Then we look at the 2nd decimal place (the 'special' digit) – it's a '9'. So we need to increase the value of the first decimal place by 1 – but it's already a '9'. So it becomes a 0 and we have to increase the value of the digit just to its left. The digit just to its left is a '9' too, so it becomes a 0, and we increase the value of the digit just to its left by 1. Here's the whole process:

Original number:

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9.99996

Writing it with no decimal places:

9

The ‘special’ digit in the original number is a ‘9’:

9.99996

So need to round up the first decimal place:

9.9

The first decimal place becomes a ‘0’, and we need to increase the value of the digit to its left by one:

9.9

For the next step it might make it easier if you remember that 9.0 is the same as 09.0. It’s already a ‘9’ as well, so it becomes a ‘0’, and we increase the value of the digit just to its left by one. The digit just to its left isn’t normally written, but if you did write it, it would be a ‘0’. So we increase it to ‘1’:

10.0

And there we have our answer, 9.99996 rounded to 1 decimal place is 10.0. The answer looks quite different to the original number doesn’t it? In this case this doesn’t mean we’re wrong however, it just shows how different the answer can look after rounding.

Rounding to a number of significant figures

Rounding off to a certain number of significant figures means that the number you write only has a certain number of digits, left *or* right of the decimal point.

What are significant digits?

A digit is counted as a significant digit *except* in the case of zeros directly after a decimal point for a number that is smaller than 1. For instance, look at the following numbers:

- 427 has 3 significant digits
- 0.5234 has 4 significant digits

- 232.03 has 5 significant digits

0.000023 only has 2 significant digits. We don't count the zeros directly after the decimal point because the number is smaller than 1. Be careful though:

23.000023 has 8 significant figures. Although it has zeros directly after the decimal point, they are counted as significant figures in this case because the number is larger than 1.

The procedure for rounding a number to a certain number of significant figures is similar to rounding to a number of decimal places:

Handy Hint #7 - Rounding to significant figures

Write the number down, but with one less digit than you've been asked to round to.

Now find the 'special' digit. The 'special' digit is the digit located just to the right of the last significant digit.

Look at the value of the 'special' digit.

If the 'special' digit is 0, 1, 2, 3, or 4, then just write your last significant digit as it is.

If the 'special' digit is a 5 or larger, you need to add one to your last significant figure.

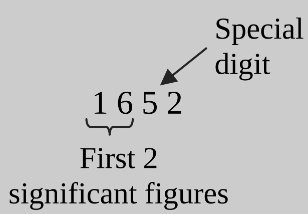
Let's do an example to show how the steps work:

Significant figures question

Round 1652 to two significant figures.

Solution

Here's a diagram pointing out each part of the number.



First we write down the number, but with only one significant figure (one less than we've been asked for):

1652

The rest of the number has been shown in a very light colour. Next we look at the value of the 'special' digit – in this case the special digit is a '5'. Because 5 is larger than 4, we need to add one to the last significant digit:

1752

And lastly, we can write the rest of the *insignificant* digits as zeros:

1700

Now, if someone looks at what you've just written, they don't know whether you've written 1700 *exactly*, or whether you've rounded some other number and it has become 1700 through rounding (like what we just did). There is a way to tell the reader how many significant figures you have written. What you need to do is use *exponential notation*. Instead of writing 1700, you'd write:

$$1.7\times10^3$$

or

$$17\times10^2$$

If you haven't done *powers* yet, for the moment just think of 10^2 as being two 10s multiplied by each other, so 17×10^2 is really 17 × 10 × 10 = 17 × 100 = 1700. In the same way, 10^3 is really three lots of 10s multiplied by each other, so 1.7×10^3 = 1.7 × 10 × 10 × 10 = 1.7 × 1000 = 1700.

This would show the reader that there are only two significant figures in your answer – the '1' and the '7'. Otherwise the reader might think that you had four significant figures – the '1', the '7' and the two 0s.

Here's another significant figures question:

Significant figures question

Round 68.5389 to 4 significant figures.

Solution

Well, I start by writing the number with only 3 significant figures (one less than I've been asked for):

68.5389

Next, I look at the value of the special digit:

68.5389

It's an 8, so I need to add one to the last significant digit:

68.54 (instead of 68.53)